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**LETTER TO THE EDITOR**

# Evolution of Gaussian wave packet and nonadiabatic geometrical phase for the time-dependent singular oscillator

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**Abstract**
We investigate the dynamics and geometric phases of a time-dependent singular oscillator. We construct certain Gaussian wave packet solutions of the corresponding Schrödinger equation, relate the latter with the classical equation of motion and explore the relationship between the associated quantum and phase angles. It is shown by a simple geometrical approach that the geometrical phase is connected with the classical nonadiabatic Hannay angle of the generalized harmonic oscillator. Our geometric approach is based on a rule for a 'natural transport' of the complex two-dimensional vector in the phase space and the results obtained are quite suggestive of similarities to the quantum mechanical two-state evolution.

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Explicitly time-dependent problems present special difficulties in classical and quantum mechanics. However, they deserve detailed study because very interesting properties emerge when, even for simple linear systems, some parameters are allowed to vary with time. For instance, particular recent interest has been devoted to systems in which evolution originates geometric contributions [1–6]. One of these, the generalized harmonic oscillator, has invoked much attention to study the nonadiabatic geometric phase for various quantum states, such as Gaussian, number, squeezed or coherent states, which can be found exactly [7–10]. Recently, the geometric phase for a cyclic wave packet solution of the generalized harmonic oscillator and its relation to Hannay's angle were studied by Ge and Ghild [7]. They introduce the time-dependent Heller Gaussian wave packet form [11]

$$\Phi(x,t) = \exp(\hbar^{-1}[-\alpha(x-q)^2 + \mathrm{i}p(x-q) + k]) \tag{1}$$

centred around the classical guiding trajectory $(q, p)$, and proceed to derive equations of motion for the complex or real parameters ($\alpha(t)$, $q(t)$, $p(t)$ and $k(t)$) which serve to specify a complete quantum wave packet.

On the other hand, the number of exactly solvable quantum time-dependent problems is very restricted; one of the rare examples admitting exact solutions of the Schrödinger equation and have been studied intensively lately [12–20] is the quantum time-dependent generalized singular oscillator

$$H = \frac{1}{2}\left[Z(t)p_x^2 + Y(t)(p_x x + x p_x) + X(t)x^2 + \frac{Z(t)l^2}{x^2}\right] \tag{2}$$

where $x$ and $p_x = -\imath\hbar\partial/\partial x$ are the quantum operators, $X(t)$, $Y(t)$ and $Z(t)$ are an arbitrary function of time and $l$ is an arbitrary constant which could be zero. The only known solution of this model are given by the Laguerre functions [15–20]. A distinguished role of the Hamiltonian (2) is explained by the fact that, in a sense, it belongs to a boundary between linear and nonlinear problems of classical and quantum mechanics. For this reason, it was used in many applications in different areas of physics. For example, it served as an initial point in constructing interesting exactly solvable models of interacting $N$-body systems [12, 13]. It was also used for modelling diatomic and polyatomic molecules [14]. It can have some relation to the problem of the relative motion of ions in electromagnetic traps [19]. Several types of exact solutions were also constructed [20].

The purpose of this letter is twofold: first, we elaborate on the dynamics of the wave packet in the time-dependent harmonic oscillator with an inverse-square potential. We summarize a number of results for wave packet dynamics and show that the time evolution can be described in terms of classical concept; in the sense that the parameters of the wavefunction evolve according to classical mechanics. We would, however, like to draw attention to the new results: the analytical expression for the Gaussian wavefunction is obtained. Second, since we are able to determine the canonical variables and the explicit form of the equations of motion, we take this opportunity to decompose the time-dependent quantum global phase factor into its geometric and dynamical parts. From the derived expression we observe that the geometrical part can be interpreted as a 'natural transport' for a family of (homothetical centred) ellipses and this is an important new result of this letter.

Since we are interested in the most general wave packet solution to the problem, we make the ansatz

$$\Psi_l(x,t) = x^{(1/2-\sqrt{(l/\hbar)^2+1/4})}\exp\left\{\frac{1}{\hbar}\left(\tfrac{1}{2}(l+\mathrm{i}pq)\left[\left(\frac{x-q}{q}\right)^2 + 2\left(\frac{x-q}{q}\right)\right] + k\right)\right\} \tag{3}$$

where $\Psi_l(x,t)$ is given as the product of a squeezed Gaussian wave packet of type (1) and a function $x$ of order $\left(1/2 - \sqrt{(l/\hbar)^2 + 1/4}\right)$, and where $q(t)$, $p(t)$, $k(t)$ are auxiliary functions of time, to be determined in what follows. This wavefunction is not normalizable for general $l$. However, it is for particular values of $l \in ]-\sqrt{3}/2, \sqrt{3}/2[$ (see the Gaussian wave packet $\Psi_l(x,t)$ below). Morever, for these geometrical phase studies, the wave packets do not have to be normalizable for general $l$. When $l = 0$, equation (3) with the $(-)$ sign (for the power of $x$) will reduce to a refined version of the standard semiclassical technique [7, 11], namely the Gaussian wave packet (1) solution of the generalized harmonic oscillator.

Inserting equation (3) into the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\Psi_l}{\partial t} = H\Psi_l \tag{4}$$

and then comparing the coefficients of various powers of $(x-q)$, leads to

$$(x-q)^2: \quad \mathrm{i}\dot{\beta} = 2Z\beta^2 - 2\mathrm{i}Y\beta + \frac{X}{2} \tag{5}$$

where $\beta = -\mathrm{i}p/(2q) - l/(2q^2)$,

$$(x-q)^1: \quad -2\mathrm{i}\left(\beta - \frac{l}{2q^2}\right)(\dot{q} - Zp - Yq) + \left(\dot{p} + Xq + Yp - \frac{Zl^2}{q^3}\right) = 0 \tag{6}$$

$$(x-q)^0: \quad p\dot{q} + \mathrm{i}\dot{k} = \frac{1}{2}\left[Zp^2 + 2Ypq + Xq^2 + \frac{Zl^2}{q^2}\right] - \frac{Z(t)l^2}{q^2}$$
$$+ 2\hbar\left(1/2 - \sqrt{(l/\hbar)^2 + 1/4}\right)\left(Z\beta - \mathrm{i}\frac{Y}{2}\right). \tag{7}$$

The $(x-q)^2$ condition determines $\beta$ by a nonlinear equation of the Riccati form, which can be transformed to a linear system by introducing a two-dimensional vector $\vec{V}^T \equiv (Q, P)$ and

$$\beta \equiv -\frac{\mathrm{i}}{2}\frac{P}{Q} \tag{8}$$

where $Q$ and $P$ may be complex. In order that $\beta$ satisfies (5), it is sufficient that $\vec{V}$ obey the classical equation

$$\dot{\vec{V}} = \begin{pmatrix}\dot{Q}\\ \dot{P}\end{pmatrix} = \begin{pmatrix} Y & Z \\ -X & -Y\end{pmatrix}\begin{pmatrix}Q\\ P\end{pmatrix} = -\mathcal{H}\vec{V}. \tag{9}$$

The $(x-q)^1$ condition makes sense if

$$\dot{q} = Zp + Yq \qquad \dot{p} = -Xq - Yp + \frac{Zl^2}{q^3} \tag{10}$$

and determines the complex guiding trajectory associated with the classical Hamiltonian

$$H(q, p, t) = \frac{1}{2}\left[Z(t)p^2 + Y(t)(pq + qp) + X(t)q^2 + \frac{Z(t)l^2}{q^2}\right]. \tag{11}$$

The $(x-q)^0$ condition determines the time-dependent global phase and normalization included in $k$ which can be rearranged in the form

$$k(t) - k(0) = \mathrm{i}\int_0^t \left(L(t') + \frac{Z(t')l^2}{q^2} - 2\hbar\left(1 - \sqrt{(l/\hbar)^2 + 1/4}\right)(Z\beta - \mathrm{i}Y/2)\right)\mathrm{d}t' \tag{12}$$

where $L(t) = p(q, \dot{q})\dot{q} - H[q, p(q, \dot{q}), t]$. Examining the three terms in the expression (12) for $k$, we see that the first two give $\mathrm{i}\{(pq) - p(0)q(0)\}/2$. The remaining term, $\gamma_l(t) = -2\left(1 - \sqrt{(l/\hbar)^2 + 1/4}\right)\int_0^t (Z\beta - \mathrm{i}Y/2)\,\mathrm{d}t'$, is the phase factor accumulated in the nonadiabatic evolution.

At this level, we can consider that the quantum problem is completely solved and the solution (3) can be rewritten as a simple wave packet

$$\Psi_l(x, t) = x^{(1/2 - \sqrt{(l/\hbar)^2 + 1/4})}\exp\left(\frac{1}{\hbar}\{-\beta(t)x^2 + \beta(0)q^2(0) + k(0) + \mathrm{i}\gamma_l(t)\}\right)$$

which is also a solution to the Schrödinger equation for the generalized singular oscillator. The reader can easily guess that the time-dependent global phase factor, namely $2\int_0^t (Z\beta - \mathrm{i}Y/2)\,\mathrm{d}t'$, can be decomposed into a dynamical part and a geometrical one. From the phase-corrected wavefunction $\xi_l(x, t) = x^{(1/2 - \sqrt{(l/\hbar)^2 + 1/4})}\exp(-\beta(t)x^2/\hbar)$, which is not the solution to the Schrödinger equation, one can in principle follow the usual method to calculate the geometrical phase.

In what follows we identify the geometrical phase factor in terms of the classical angle $\theta_H(t)$ (Hannay's angle), and we obtain a new analytical expression for the geometrical phase factor based only on a simple geometrical approach in the phase space description. This treatment allows one to view the classical–quantum correspondence in a new light and yields a novel classical interpretation of quantum phenomena. Before that, we derive the geometrical part by confirming whether there exists a natural transport (with respect to the symplectic structure) for a family of (homothetical centred) ellipses, and finally, we re-establish the connection between the quantum phase and the classical angle.

In the literature, $\theta_H(t)$ is usually defined in relation to the introduction of time-dependent canonical transformations. However, a simple geometrical approach (hence not calculational) may be formulated to deduce the decomposition of the total phase factor (12). It is well known (see, for example, [1, 2]) that Hannay's angle is defined when the closed curves of constant action variables return to the original curves after a time evolution. In this letter we are interested in the infinitesimal transport, thus the cyclic properties of the closed curves of constant action variables will be omitted in the following.

Let us consider the classical equation (9) which allows a geometrical interpretation of the evolution in phase space. The main property of this evolution is that it is linear and area preserving. This implies that any initial conditions at $t = 0$ on a centred ellipse $\mathcal{E}(0)$ in phase space evolve at time $t$ on a similar ellipse $\mathcal{E}(t)$ of the same area. A little thought shows that, more precisely, two points on $\mathcal{E}(0)$ whose parameters differ by $\Delta\varphi$ evolve in points on $\mathcal{E}(t)$ with the same difference of parameter. The reason is that the standard parameter $\varphi$ ($\varphi \in [0, 2\pi]$) which parametrizes a point M on an ellipse is such that it is proportional to the area swept by the vector $\vec{v} = \overrightarrow{OM}$.

Analytically, let $\vec{E}$ be a complex two-dimensional vector; it is known (for instance from optics) that one can describe (homothetical centred) ellipses as the set of vectors $\mathrm{Re}[A\,\mathrm{e}^{-\mathrm{i}\varphi}\vec{E}(t)]$. The natural origins of these ellipses are the points associated with $\varphi = 0$. $A$ and $\varphi$ may be considered as the ('action-angle') coordinates of a point in phase space, with respect to the family associated with $\vec{E}(t)$. However, a more geometrical approach may be formulated as follows: is there a transport from the family of ellipses associated with $\vec{E}(0)$ to that associated with $\vec{E}(t)$ which is natural with respect to the symplectic structure in phase space? Clearly, this transport must preserve area. This implies that $\vec{E}^* \wedge \vec{E}$ is kept fixed, i.e. for an infinitesimal transport $\delta\vec{E}$: $\mathrm{Im}(\vec{E}^* \wedge \delta\vec{E}) = 0$. But this is not sufficient since it remains to give precisely how one point (for example, the origin) is transported: for this, one simply requires that the area which is swept by the vectors $\vec{v}(\varphi)$ (on an ellipse) during the transport has, when averaged over $\varphi$, a mean value equal to zero; this implies $\mathrm{Re}(\vec{E}^* \wedge \delta\vec{E}) = 0$. Therefore, the transport is defined by

$$\vec{E}^* \wedge \delta\vec{E} = 0. \tag{13}$$

Hannay's angle is then obtained by setting $\vec{E} + \delta\vec{E} = (\vec{E} + \mathrm{d}\vec{E})\exp(-\mathrm{i}\mathrm{d}\theta_H)$, as

$$\dot{\theta}_H = \frac{\vec{E}^* \wedge \dot{\vec{E}}}{\mathrm{i}(\vec{E}^* \wedge \vec{E})}. \tag{14}$$

Obviously equation (14) can be considered as a new result for the nonadiabatic Hannay angle of the generalized harmonic oscillator.

Within such a formalism the above remarks justify that the general solution of equation (9) may be looked for in the form

$$\vec{V}(t) = A\,\mathrm{e}^{-\mathrm{i}(\theta(t)+\varphi)}\vec{E}(t) \qquad (\theta(0) = 0) \tag{15}$$

with $\mathrm{i}\vec{E}^* \wedge \vec{E}$ conserved and $A$ and $\varphi$ fixed ($A$ and $\varphi$ are the conditions measured with respect to the family $\vec{E}(0)$). Postponing the precise calculation of $\theta(t)$ and $\vec{E}(t)$, some interesting remarks can be deduced from the formula (15). The first one is that the quadratic quantity in phase spaces $q$ and $p$,

$$I = |\vec{v} \wedge \vec{E}|^2 \qquad (\vec{v} = \mathrm{Re}[\vec{V}(t) = A\,\mathrm{e}^{-\mathrm{i}(\theta(t)+\varphi)}\vec{E}(t)]) \tag{16$a$}$$

is invariant ($\vec{E} \wedge \vec{E} = 0$ and $\vec{E}^* \wedge \vec{E}$ constant) and is known as the Lewis invariant. The second one is that $\theta(t)$ which defines an angular drift of the origin points of $\vec{E}(0)$ (measured with respect to $\vec{E}(t)$) is naturally decomposed into a geometrical part (Hannay's angle) and a dynamical one,

$$\dot{\theta} = \frac{\vec{E}^* \wedge \dot{\vec{E}}}{\mathrm{i}(\vec{E}^* \wedge \vec{E})} + \frac{\vec{E}^* \wedge \mathcal{H}\vec{E}}{\mathrm{i}(\vec{E}^* \wedge \vec{E})}. \tag{16$b$}$$

(This relation is obtained by inserting $\vec{V}(t)$ in classical equation (9) and making the wedge product with $\vec{V}^*(t)$.)

If one wants to explicitly calculate $\dot{\theta}_H$ and $\dot{\theta}$, it will be convenient to normalize $\vec{E}$ such that $-\mathrm{i}(\vec{E}^* \wedge \vec{E}) = 4I$ (which fixes the ellipse area), and to parametrize the ellipse as

$$\vec{E}(t) = \begin{pmatrix} \sqrt{QQ^*} \\ 2\mathrm{i}\beta(t)\sqrt{QQ^*} \end{pmatrix} \tag{17}$$

where the first component of $\vec{E}$ is taken to be real. Then, one can find, making use of (14) and (16$b$), that the Hannay angle is

$$\dot{\theta}_H = -\mathrm{i}\frac{\dot{\beta}}{\beta + \beta^*} - \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln(QQ^*) \tag{18}$$

and the total angle is

$$\dot{\theta} = -2(Z\beta - \mathrm{i}Y/2) - \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln(QQ^*). \tag{19}$$

The choice of $\vec{E}(t)$, equation (17), corresponds to the parametrization of the ellipse in phase space $(q, p)$ as

$$\begin{aligned} q &= A\sqrt{QQ^*}\cos\theta \\ p &= A\sqrt{QQ^*}\{\mathrm{i}(\beta - \beta^*)\cos\theta + (\beta + \beta^*)\sin\theta\} \end{aligned} \tag{20}$$

and to the quadratic invariant equation (16$a$)

$$I = PP^*q^2 - (PQ^* + P^*Q)pq + QQ^*p^2 \tag{21}$$

associated with the generalized harmonic oscillator $H_{\mathrm{GHO}} = \frac{1}{2}[Z(t)p^2 + 2Y(t)pq + X(t)q^2]$.

On comparing equation (19) and the time-dependent phase factor $\gamma_l(t)$ (equation (12)), we see that $\gamma_l(t)$ is the quantum counterpart of $\theta(t)$ (equation (19)),

$$\gamma_l(t) = (1 - \sqrt{(l/\hbar)^2 + 1/4})\left[\theta(t) + \frac{\mathrm{i}}{2}\ln\left\{\frac{Q(t)Q^*(t)}{Q(0)Q^*(0)}\right\}\right] \tag{22}$$

where the logarithm term goes 'downstairs' as the time-dependent normalization factor in $\Psi_l(x, t)$. The remaining term in $\gamma_l(t)$ is recognized as the phase factors acquired by the wave packet in its evolution and it is the quantum counterpart of equation (19). This establishes a useful relationship between the associated quantum phase of the generalized singular oscillator and the classical angle of the generalized harmonic oscillator.

Then, we can reach a simple relation between the geometrical phase for the quantum singular oscillator and the nonadiabatic Hannay angle associated with the generalized harmonic oscillator

$$\dot{\gamma}_l^G(t) = -\left(1 - \sqrt{(l/\hbar)^2 + 1/4}\right)\frac{\vec{E}^* \wedge \dot{\vec{E}}}{4I} \tag{23}$$

where the first part is independent of $\hbar$ and is equal to the Hannay angle of the generalized harmonic oscillator, the second part depends on $\hbar$ and $l$.

In conclusion, we have constructed certain Gaussian wave packet solutions of a time-dependent Schrödinger equation for the singular oscillator, related the latter to the classical equation of motion and explored the relationship between the associated quantum phases and classical angles of the generalized harmonic oscillator. The classical version of the generalized harmonic oscillator has been discussed, and a new expression for the nonadiabatic Hannay angle has been obtained by confirming whether there exists a natural transport (with respect to the symplectic structure) for a family of (homothetical centred) ellipses. The invariant associated with the generalized harmonic oscillator is deduced in a simple way. When the parameter $l$ vanishes, we see that the Gaussian wave packet $\Psi_l(x, t)$ reduces to a refined version of the standard semiclassical technique which corresponds to the evolution of the 'ground' state of the time-dependent generalized harmonic oscillator, and the geometrical phase is equal to one-half of the classical geometrical angle. This is just what was obtained in [7].

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